Algebraic IDA-PBC for Polynomial Systems with Input Saturation: An SOS-based Approach

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Abstract: The necessity to deal with partial differential equations (PDEs) and the dissipation condition are the main adversities in the application of Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC). Recently, an algebraic solution of IDA-PBC has been explored for a class of affine polynomial systems by using sum of squares (SOS) and semidefinite programming (SDP). In this work, we extend the previous method by incorporating actuator saturation (AS) and two minimization objectives in the SDP. Our results are validated on two polynomial systems.

Keywords: Port-Hamiltonian Systems, IDA-PBC, Polynomial Systems, Sum of Squares, Actuator Saturation.

IDA-PBC Algebraico Para Sistemas Polinomiales con Saturación en las Entradas: Un Enfoque Basado en SOS

Resumen: La solución de ecuaciones diferenciales parciales (PDE) y la condición de disipación son las principales adversidades en la aplicación de *Interconnection and Damping Assignment Passivity-Based Control* (IDA-PBC). Recientemente, se ha explorado una solución algebraica de IDA-PBC para una clase de sistemas polinomiales utilizando el método de suma de cuadrados (SOS) y la programación semidefinida (SDP). En este trabajo se amplía el método anterior incorporando saturación en los actuadores (AS) y dos objetivos de minimización en la SDP. Nuestros resultados son validados en dos sistemas polinómicos.

Palabras claves: Sistemas Port-Hamiltonianos, IDA-PBC, Sistemas Polinomiales, Suma de Cuadrados, Saturación del Actuador.

1. INTRODUCTION

Over the last decade, Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) has experienced increasing practice due its wide applicability (Petrović et al., 2001; Batlle et al., 2004, 2007; Ortega and García-Canseco, 2004; Li et al., 2010; Astolfi and Ortega, 2001; Fujimoto et al., 2001; Renton et al., 2012; Li et al., 2013; Astolfi et al., 2002a; Xue and Zhiyong, 2017). The standard IDA-PBC method requires a two step procedure: energy shaping and damping injection. The first depends on the solution of partial differential equations (PDEs) and the second together with zero state detectability (ZSD) of the closed-loop guarantees asymptotic stability.

In order to simplify the PDE problem Viola et al. (2007) have introduced a change of coordinates and a modification of the target dynamics. With the objective to completely avoid PDEs, the following leading methods have been proposed: constructive procedures (Donaire et al., 2016a; Borja et al., 2016; Romero et al., 2017), implicit port-Hamiltonian representation (Macchelli, 2014; Castaños and Gromov, 2016) and an algebraic approach (Fujimoto and Sugie, 2001; Batlle et al., 2007; Nunna et al., 2015). In addition, it has been shown in (Batlle et al., 2007; Donaire et al., 2016b) that a two step IDA-PBC may be restrictive in some cases, thus introducing a single step procedure (SIDA-PBC). Furthermore, dissipation in the under-actuated degrees of freedom, see (Gómez-Estern and Van der Schaft, 2004), may also turn out an obstacle for the implementation of IDA-PBC on some systems, e.g. on the cart-pole system (Delgado and Kotyczka, 2014).

It is well-known that actuator saturation (AS) can cause performance losses or even lead to closed-loop instability. In this context, Åström et al. (2008); Escobar et al. (1999) have studied PBC with AS on two specific systems. Sun et al. (2009); Wei and Yuzhen (2010) analyze stability for saturation in the damping injection term. A variable structure approach to energy shaping for a class of Port-Hamiltonians system is developed in (Macchelli, 2002; Macchelli et al., 2003). Besides, Sprangers et al. (2015) studied a reinforcement learning method for energy shaping which shows robust properties under AS.

In polynomial systems the sum of squares (SOS) approach with semidefinite programming (SDP) allows to synthesize Lyapunov functions (Parrilo, 2000), optimal (Ichihara, 2009), robust (Zhu et al., 2018; Jennawasin et al., 2010), fuzzy (Wibowo et al., 2014;

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Yu and Wang, 2013) and AS-focused controllers (Jennawasin et al., 2012; Valmorbida et al., 2013; Ichihara, 2013), among others. This AS controllers, calculated with SOS, use the polytope representation introduced in (Hu and Lin, 2001) for linear system with multiple input saturation.

For a class of polynomial affine systems, lately, Cieza and Reger (2018) have presented an algebraic method the conditions of which are met by means of SOS and SDP solutions. The method solves the typical problems of IDA-PBC at the expense of an adequate parametrization and selection of the Hamiltonian. To the best knowledge of the authors there is no definitive solution to the AS controller design problem with IDA-PBC. The underlying work shall now also incorporate AS and two minimization objectives in the SDP solver, extending the algorithm of Cieza and Reger (2018).

The work is organized as follows. We summarize the concepts of IDA-PBC for nonlinear affine systems in Section 2. Section 3 recapitulates the algebraic method of Cieza and Reger (2018). In Section 4 we solve the AS problem in the algebraic approach using additionally two minimization objectives. We discuss the application of SOS methodology and verify our results in Section 5, applying the approach on two polynomial systems. Finally, we draw our conclusions in Section 6.

2. IDA-PBC for Affine Nonlinear Systems

Let us recall the IDA-PBC approach for nonlinear affine systems introduced by Ortega and García-Canseco (2004). Consider the system

$$\dot{x} = f(x) + g(x)\,u \tag{1}$$

and the target port-Hamiltonian system

$$\dot{x} = F_d(x)\left(\frac{\partial H_d}{\partial x}\right)^{\top}(x), \tag{2}$$

where $\mathscr{X}_h \subset \mathbb{R}^{n_x}$ is the state space manifold, $\mathscr{U} \subset \mathbb{R}^m$ is the input space, g has full rank, the skew symmetric portion $\frac{1}{2}\left(F_d - F_d^{\top}\right) \in \mathbb{R}^{n_x \times n_x}$ is the interconnection matrix, the symmetric portion $\mathbb{R}^{n_x \times n_x} \ni \frac{1}{2}\left(F_d + F_d^{\top}\right) \preceq 0$ is the dissipation matrix and $H_d : \mathscr{X}_h \to \mathbb{R}$ with $x^{\star} = \operatorname{argmin} H_d(x)$ is the desired positive definite Hamiltonian. If the matching condition[1]

$$g_{\perp}(x) f(x) = g_{\perp}(x) F_d(x)\left(\frac{\partial H_d}{\partial x}\right)^{\top}(x) \tag{3}$$

is fulfilled for some F_d, H_d and full rank left annihilator[2] $g_{\perp}$ then the control law

$$u_I = \left(g^{\top}(x) g(x)\right)^{-1} g^{\top}(x)\left(F_d(x)\left(\frac{\partial H_d}{\partial x}\right)^{\top}(x) - f(x)\right)$$

transforms (1) into the stable system (2) with Lyapunov function H_d. Asymptotic stability of $x^{\star}$ in the attainable set $\mathscr{X}_a = \{x \in \mathscr{X}_h \mid g_{\perp} f = 0\}$ may be demonstrated e.g. by using passivity with Lyapunov stability theory, see (Astolfi et al., 2002b; Sepulchre et al., 1997).

[1] If in the matching condition (3) the structure of $F_d(x)$ is fixed, then (3) is a PDE. If $H(x)$ is fixed, then (3) is an algebraic equation.

[2] The full rank left annihilator $g_{\perp}$ is given by $g_{\perp}(x)g(x) = 0$ and $\operatorname{rank}(g_{\perp}) = n_x - m$. Consequently, (3) exists iff $n_x > m$.

3. IDA-PBC for Polynomial Systems

In this section we summarize Proposition 4–5 from (Cieza and Reger, 2018) for $\beta = 1$. The variable β of the aforementioned work can be considered as a scaling factor, which does not alter our main results.

3.1 Algebraic IDA-PBC

Let $\Gamma(x)$, $g(x)$, $g_{\perp}(x)$, and $g_{\perp}(x)f(x)$ be polynomial functions in

$$\Gamma(x)\dot{x} = f(x) + g(x)u \tag{4}$$

and the desired closed loop port-Hamiltonian system

$$\Gamma(x)\dot{x} = \begin{bmatrix} g_{\perp}(x) \\ g^{\top}(x) \end{bmatrix}^{-1} F(x)\left(\frac{\partial H_d}{\partial x}\right)^{\top}(x), \tag{5}$$

where

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad H_d(x) = z^{\top}(x) P^{-1} z(x), \quad P = P^{\top} \succ 0.$$

Without loss of generality let $\operatorname{argmin} H_d(x) = 0$. Besides, state and input spaces remain as in (1), $z : \mathscr{X}_h \to \mathbb{R}^{n_z}$ is a vector of polynomials with $n_z \geq n_x$, $P \in \mathbb{R}^{n_z \times n_z}$ is a constant matrix, $F, \Gamma : \mathscr{X}_h \to \mathbb{R}^{n_x \times n_x}$ are full rank polynomial matrices[3], and F holds the portions $F_2(x) \in \mathbb{R}^{m \times n_x}$ and $F_1(x) \in \mathbb{R}^{n_x - m \times n_x}$.

Proposition 1 *Closing the loop of system* (4) *with control*

$$u_b = (g^{\top} g)^{-1}\left(F_2\left(\frac{\partial z}{\partial x}\right)^{\top} P^{-1} z - g^{\top} f\right) \tag{6}$$

renders the equilibrium point $x^{\star} = 0$ of the closed loop system (5) *locally stable for any initial state $x(0) = x_0 \in \mathscr{X}_P = \{x \in \mathbb{R}^{n_x} \mid H_d(x) = z^{\top}(x) P^{-1} z(x) \leq 1\}$ whenever the following conditions are fulfilled for all $x \in \mathscr{X}_h = \{x \in \mathbb{R}^{n_x} \mid 0 \leq h(x) = 1 - z^{\top}(x) S_h^{-1} z(x), S_h \succ 0\}$:*

- *C1 There exist polynomial functions $\Lambda_1(x)$, $g_{\perp}(x)$ and $z(x)$ such that $g_{\perp}(x) f(x) = \Lambda_1(x) z(x)$ and $z(x) = 0$ iff $x = 0$. Besides, if $n_z = n_x$ then $\frac{\partial z}{\partial x}$ is unimodular[4], else $\frac{\partial z}{\partial x}$ has rank n_x and $\left(\frac{\partial z}{\partial x}\right)_{\perp}(x) P \Lambda_1^{\top}(x) = 0$ with $\left(\frac{\partial z}{\partial x}\right)_{\perp}$ the full rank left annihilator of the Jacobian of z.*
- *C2 There is a polynomial function $N(x)$ such that if $n_z = n_x$ then $N(x) = \left(\frac{\partial z}{\partial x}\right)^{-\top}(x)$, else $\left(\frac{\partial z}{\partial x}\right)^{\top}(x) N(x) = I_{n_x}$ and $\left[N \ \left(\frac{\partial z}{\partial x}\right)^{\top}_{\perp}\right]$ is a full rank square matrix.*[5]

[3] Invertibility of Γ enables (4) to take the usual form $\dot{x} = \bar{f}(x) + \bar{g}(x)u$ and $F_d(x) = \Gamma^{-1}(x)\begin{bmatrix} g_{\perp}(x) \\ g^{\top}(x) \end{bmatrix}^{-1} F(x)$.

[4] Polynomial matrices are unimodular if the inverse matrix again is a polynomial matrix. Their determinant always is a non-zero constant.

[5] Here I_n represents the identity matrix of size n.

C3 There is a constant matrix P, and polynomial matrices $0 \preceq S_1(x) \in \mathbb{R}^{n_x \times n_x}$ *and* $F_2(x)$ *such that*

$$-\Phi(x) - \Phi^\top(x) - h(x)S_1(x) \succeq 0, \tag{7}$$

$$S_h \succeq P \succeq z(x_0)z^\top(x_0), \tag{8}$$

$$\Phi = \begin{bmatrix} g_\perp \\ g^\top \end{bmatrix} \Gamma \begin{bmatrix} F_1^\top & F_2^\top \end{bmatrix}, \qquad F_1(x) = \Lambda_1(x)PN(x).$$

Furthermore, the origin of (5) *is asymptotically stable if*

$$-\Phi(x) - \Phi^\top(x) - h(x)S_1(x) \succ 0 \quad \textit{and} \tag{9}$$

for $n_z > n_x$, $\left(\frac{\partial z}{\partial x}\right)^\top(x)P^{-1}z(x) = 0$ *implies* $x = 0$.

Proof 1 *It can be found in (Cieza and Reger, 2018) with a modification of* (8) *using the Schur complement.* ■

3.2 Existence of u_b

The next proposition provides a sufficient condition for the existence of F_2, i.e. the existence of the (asymptotically) stabilizing control law (6).

Proposition 2 *Consider* $x_0 \in \mathscr{X}_P \subset \mathscr{X}_h$ *and assume* Λ_1, $g_\perp$, z, N *are selected according to C1 and C2. Then there exists a function* $F_2(x)$ *that meets C3 if there exists* $P = P^\top \succ 0$ *and* $S_2(x) \succeq 0$ *such that*

$$-\phi(x) - \phi^\top(x) - h(x)S_2(x) \succeq 0, \tag{10}$$

$$\phi(x) = g_\perp(x)\Gamma(x)F_1^\top(x) = g_\perp(x)\Gamma(x)N^\top(x)P\Lambda_1^\top(x),$$

with $\phi(x)$, $S_2(x) \in \mathbb{R}^{n_x - m \times n_x - m}$. *Additionally,* (9) *is solvable as long as*

$$-\phi(x) - \phi^\top(x) - h(x)S_2(x) \succ 0. \tag{11}$$

Lastly, if (10) *or* (11) *are satisfied, then a solution for* F_2 *with* $0 \succ L(x) + L^\top(x) \in \mathbb{R}^{m \times m}$ *is*

$$F_2 = \begin{bmatrix} -g^\top(x)\Gamma(x)F_1^\top(x), & L(x) \end{bmatrix} \begin{bmatrix} g_\perp(x) \\ g^\top(x) \end{bmatrix}^{-\top} \Gamma^{-\top}(x) \tag{12}$$

Proof 2 *See (Cieza and Reger, 2018, Prop. 5).* ■

Application of the algorithm starts with adequate selection of Λ_1, z, $g_\perp$ and N according to (4), (5), C1 and C2. Later, we choose h and solve (for convenience) (7), (9), (10) or (11) searching for P (and F_2 in case of Prop. 1) under (8) which defines an upper and lower bound on P, namely $x_0 \in \mathscr{X}_P \subset \mathscr{X}_h$ for some given x_0.

In comparison, Proposition 2 requires the solution of smaller LMIs to calculate the parametrized function F_2, see (12), or to guarantee its existence, whereas Prop. 1 defines F_2 as general function s.t. (7) (and (9)) is satisfied, which grants more flexibility at the expense of computational cost. In order to use SOS with SDP we force F_2 (in Prop. 1) to be a polynomial function of some selected degree.

Proposition 2 can also be used as a fast indicator such that Proposition 1 will work. Note that Proposition 2 contains the minimal conditions that P, Λ_1, $g_\perp$, N, h and Γ have to satisfy, and it guarantees the existence of a not necessarily polynomial function F_2. Hence, if we constrain F_2 to be polynomial, then Proposition 2 is experimentally still a good, but not an unconditionally reliable reference.

4. Main Result

In view of Proposition 1 and 2, we extend the results of Cieza and Reger (2018) to consider actuator saturation (AS). In addition, we define two possible minimizations (optimization objectives) for the SDP.

4.1 Actuator Saturation (AS)

Proposition 3 *Let all conditions of Prop. 1 for local (asymptotic) stability be satisfied and assume:*

C5 There exist polynomial matrices $\Lambda_2(x) \in \mathbb{R}^{m \times n_z}$ *and* $0 \preceq S_3(x) \in \mathbb{R}^{m \times m}$ *such that*

$$\begin{bmatrix} \Lambda_1(x) \\ \Lambda_2(x) \end{bmatrix} z(x) = \begin{bmatrix} g_\perp(x) \\ g^\top(x) \end{bmatrix} f(x), \tag{13}$$

$$\begin{bmatrix} \eta_{11}(x) & \eta_{12}(x) \\ \eta_{12}^\top(x) & P \end{bmatrix} \succeq 0, \tag{14}$$

$$\eta_{11}(x) = (g^\top g)(x)\, S_u(x)(g^\top g)(x) - h(x)S_3(x),$$

$$\eta_{12}(x) = F_2(x)\left(\frac{\partial z}{\partial x}\right)^\top(x) - \Lambda_2(x)P.$$

Then the stabilizing control law (6) *is restricted to* $\mathscr{U}_b = \{u_b \in \mathscr{U} \mid u_b^\top S_u^{-1}(x)u_b \leq 1\}$ *with* $\mathbb{R}^{m \times m} \ni S_u(x) = S_u^\top(x) \succ 0$ *for any* $x \in \mathscr{X}_P$.

Proof 3 *Multiplying* (14) *on both sides by adequate matrices and using the Schur complement yields*

$$I_m - WW^\top \succeq hS_u^{-\frac{1}{2}}(g^\top g)^{-1}S_3(g^\top g)^{-1}S_u^{-\frac{1}{2}} \tag{15}$$

with $W(x) = S_u^{-\frac{1}{2}}(g^\top g)^{-1}(F_2\left(\frac{\partial z}{\partial x}\right)^\top P^{-\frac{1}{2}} - \Lambda_2 P^{\frac{1}{2}})$ *and* $P^{\frac{1}{2}}P^{\frac{1}{2}} = P$. *Now taking the spectral norm on* (15) *for* $x \in \mathscr{X}_P \subset \mathscr{X}_h$, *i.e.* $h \geq 0$, *and the definition of* $\mathscr{X}_P$ *as* $\left\|P^{-\frac{1}{2}}z\right\|_2^2 \leq 1$ *we have*

$$\begin{aligned} 1 &\geq \left\| S_u^{-\frac{1}{2}}(g^\top g)^{-1}\left(F_2\left(\frac{\partial z}{\partial x}\right)^\top P^{-\frac{1}{2}} - \Lambda_2 P^{\frac{1}{2}}\right)\right\|_2^2 \\ &\geq \left\| S_u^{-\frac{1}{2}}(g^\top g)^{-1}\left(F_2\left(\frac{\partial z}{\partial x}\right)^\top P^{-\frac{1}{2}} - \Lambda_2 P^{\frac{1}{2}}\right)\right\|_2^2 \left\|P^{-\frac{1}{2}}z\right\|_2^2 \\ &\geq \left\| S_u^{-\frac{1}{2}}(g^\top g)^{-1}\left(F_2\left(\frac{\partial z}{\partial x}\right)^\top P^{-1}z - \Lambda_2 z\right)\right\|_2^2 \\ &= \left\|S_u^{-\frac{1}{2}}u_b\right\|_2^2 = u_b^\top S_u^{-1}u_b, \end{aligned}$$

where last equalities are obtained with (13) *and* (6). ■

After solving the conditions of Proposition 1 and 3, we may calculate a control input $u_b \in \mathscr{U}_b$, for any $x \in \mathscr{X}_P$. Proposition 3 can also be extended to work with Prop. 2 by replacing (12) in (14). This yields an LMI which is not necessarily polynomial. Therefore, we restrict L to be polynomial and multiply (14) on the right with the square non-singular matrix[6]

$$\begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & N(x)\Gamma^\top(x)g_\perp^\top(x) & N(x)\Gamma^\top(x)g(x) & \left(\frac{\partial z}{\partial x}\right)_\perp^\top(x) \end{bmatrix}$$

and on the left by its transpose. This results in conditions that can be solved by means of SOS + SDP.

Following the works of Hu and Lin (2001); Valmorbida et al. (2013); Ichihara (2013), among others, we may use the polytope or polytopic saturation model within the algebraic IDA-PBC, as phrased in the following proposition.

Proposition 4 *Let the conditions of Propositions 1 and 3 be satisfied for some system of the form* (4) *resulting in some matrices P, F_2 and a locally (asymptotically) stabilizing constrained controller $u = u_b \in \mathscr{U}_b$ given by* (6)*. Consequently, there is a new (asymptotically) stabilizing control action*

$$u_s \in \mathscr{U}_s = \left\{ u_s = u_b + (g^\top g)^{-1}\Theta u_\delta | \theta_k \in [0,1] \right\}, \tag{16}$$

$$u_\delta = (F_{21} - F_{20}) \left(\frac{\partial z}{\partial x}\right)^\top P^{-1} z, \qquad F_2 = F_{20},$$

provided that there exist matrices $F_{21}(x) \in \mathbb{R}^{m \times n_z}$ and $\bar{S}_{i_1 \ldots i_m}(x) \succeq 0$, s.t. for all $i_k \in \{0,1\}$ with $k = 1 \ldots m$,

$$-\Phi_{i_1 \ldots i_m}(x) - \Phi_{i_1 \ldots i_m}^\top(x) - h(x)\bar{S}_{i_1 \ldots i_m}(x) \succeq 0, \tag{17}$$

$$\Phi_{i_1 \ldots i_m} = \begin{bmatrix} g_\perp \\ g^\top \end{bmatrix} \Gamma \begin{bmatrix} F_1^\top & F_{2i_1}^\top(x)e_1 & \ldots & F_{2i_m}^\top(x)e_m \end{bmatrix},$$

where $\Theta = diag(\theta_1, \ldots, \theta_m)$ and e_i the ith unity vector. In addition, asymptotic stability is achieved if (9) *are satisfied and* (17) *is strict.*

Proof 4 *Define*

$$F_2 = \Theta F_{21} + (I_m - \Theta) F_{20}, \quad \textit{(polytope)} \tag{18}$$

and $\beta_{k0} + \beta_{k1} = 1$, $\theta_k = \beta_{k0}$, then (16) *follows from* (6) *and* (18)*. Then, replace* (18) *in* (7)*, multiply it by $\sum_{i_k=0}^{1} \beta_{ki_k}$ for convenience and substitute S_1 with $\bar{S}_{i_1 \ldots i_m}$ which does not affect stability, see (Cieza and Reger, 2018). Then* (7) *results in*

$$0 \preceq \sum_{i_1=0}^{1} \beta_{1i_1} \ldots \sum_{i_m=0}^{1} \beta_{mi_m} \left(-\Phi_{i_1 \ldots i_m} - \Phi_{i_1 \ldots i_m}^\top - h\bar{S}_{i_1 \ldots i_m} \right),$$

and rewritten as a sum of positive semidefinite polynomial functions (convex set) this yields

$$0 \preceq \sum_{j=0}^{2^m - 1} \bar{\beta}_j \left(-\Phi_j - \Phi_j^\top - h\bar{S}_j \right), \tag{19}$$

with $j = \sum_{k=1}^{m} i_k 2^{k-1}$, $\sum_{j=0}^{2^m-1} \bar{\beta}_j = 1$. Therefore, a sufficient condition for (19) *is* (17)*. The proof of asymptotic stability follows a similar procedure.* ■

[6]Note that using C2, the square matrix $[N \;\; \left(\frac{\partial z}{\partial x}\right)_\perp^\top]$ and as a consequence $[N\Gamma^\top g_\perp^\top, N\Gamma^\top g, \left(\frac{\partial z}{\partial x}\right)_\perp^\top]$ has full rank.

Proposition 4 implies that if there is a solution to the conditions of Propositions 1 and 3 with (17), then there also exists an (asymptotically) stabilizing control law (16). In addition, if $F_2 = F_{20} = F_{21}$ then (16) is reduced to (6) and (17) becomes (7) (or (10)). Proposition 4 can be easily extended to work with Prop. 2 (instead of 1). In this case, (17) is reduced to

$$\begin{bmatrix} L_{i_1}^\top e_1 & \ldots & L_{i_m}^\top e_m \end{bmatrix} + \begin{bmatrix} L_{i_1}^\top e_1 & \ldots & L_{i_m}^\top e_m \end{bmatrix}^\top \prec 0,$$

$$F_{2i} = \begin{bmatrix} -g^\top(x)\Gamma(x)F_1^\top(x), & L_i(x) \end{bmatrix} \begin{bmatrix} g_\perp \\ g^\top \end{bmatrix}^{-\top} \Gamma^{-\top}(x).$$

In the same way as in (Hu and Lin, 2001; Valmorbida et al., 2013; Ichihara, 2013) for multiple input systems, we can adopt the independent input saturation given by $u_{\text{sat-i}} = \text{sat}(u_x, \bar{u}, \underline{u})$ and $u_x = u_b + (g^\top g)u_\delta$, where $\bar{u}$ and $\underline{u}$ are maximum and minimum values of u_b in $\mathscr{U}_b$. Figure 1 illustrates the situation for $m = 2$, $g^\top g = I_2$, u_x, u_b, $u_{\text{sat-i}}$ and sets $\mathscr{U}_b$, $\mathscr{U}_s$.

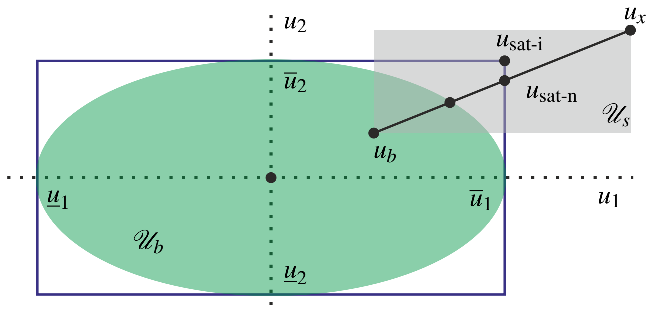

Figura 1. Relations of u constrained and saturated.

Here, we also observed that in order to have AS, independent input saturation ($u_{\text{sat-i}}$) is not the only solution. Therefore, to simplify (17), we select $\theta_1 = \theta_2 = \cdots = \theta_m$ and a new saturation function given by

$$u_{\text{sat-n}} = u_b + (g^\top g)^{-1} u_\delta \min(\rho_1, \ldots, \rho_m),$$

$$\rho_k = \begin{cases} \frac{e_k^\top(\bar{u} - u_b)}{e_k^\top(g^\top g)u_\delta}, & \text{if } e_k^\top u_x > e_k^\top \bar{u}, \quad e_k^\top(g^\top g)u_\delta \neq 0, \\ \frac{e_k^\top(\underline{u} - u_b)}{e_k^\top(g^\top g)u_\delta}, & \text{if } e_k^\top u_x < e_k^\top \underline{u}, \quad e_k^\top(g^\top g)u_\delta \neq 0, \\ 1, & \text{otherwise}, \end{cases}$$

which is also shown in Figure 1. Selection of $u_{\text{sat-n}}$ reduce 2^{m-1} inequalities and polynomial matrices $\bar{S}_\star$ in (17).

4.2 Optimization Objectives in SDP

Proposition 1–3 only guarantee a solution for P and $F_2(x)$ without any performance or optimization goal in the SDP. In addition, we may set the following simple objectives:

Optimization 1 (Volume maximization of $\mathscr{X}_P$)

$$\begin{aligned} &\textit{minimize} && trace(Y) \\ &\textit{subject to} && \begin{bmatrix} P & I_{n_z} \\ I_{n_z} & Y \end{bmatrix} \succeq 0. \end{aligned} \tag{20}$$

Proof 5 *The volume of $\mathscr{X}_P$ is proportional to $\sqrt{det(P)}$, see (Boyd et al., 1994, pp. 48-49). In addition, from KL-divergence between* *two multivariate normal distributions, we obtain the relation*

$$trace(I_{n_z} - A^{-1}) \preceq log(det(A)) \preceq trace(A - I_{n_z}) \quad (21)$$

for any real matrix $A \succ 0$. Therefore, maximizing the volume of $\mathscr{X}_P$ with $P \succ 0$ is equivalent to maximize $log(det(P))$. Using (21) *we enlarge the minimum bound of $log(det(P))$ by minimization of $trace(P^{-1})$ which is equivalent to Opt. 1 with Schur complement in* (20). ■

This minimization is also used empirically in (Ichihara, 2013). Optimization 1 maximizes the volume of $\mathscr{X}_P$ by maximizing the minimum bound of P given by Y^{-1}. Note that searching for the biggest $\mathscr{X}_P$ does not demand the explicit selection of x_0 (right hand side of (8)).

Optimization 2 (Volume minimization of $\mathscr{U}_b$)

$$\begin{aligned} &minimize \quad trace(S_u) \\ &subject\ to \quad S_u = constant. \end{aligned}$$

Proof 6 *Along the same lines of Optimization 1, except that we consider the upper bound of* (21). ■

Without loss of generality, define $F_2(x) = \bar{F}_2(x)PN(x)$, $0 = \bar{F}_2(x)P\left(\frac{\partial z}{\partial x}\right)_{\perp}^{\top}(x)$, for some function $\bar{F}_2 \in \mathbb{R}^{m\times n_z}$. Then, (14) becomes

$$(g^{\top}g)\ S_u(g^{\top}g) - (\bar{F}_2 - \Lambda_2)P(\bar{F}_2 - \Lambda_2)^{\top} \succeq 0,$$

for all $x \in \mathscr{X}_h$. This shows that minimization of S_u (upper bound of u) is equivalent to minimize $\bar{F}_2 - \Lambda_2$ and an upper bound of P. As a consequence, it is required to have at least one minimum bound on P (right hand side of (8) or Opt. 1).

5. Simulations

It is well-known that the SOS property is a sufficient condition for checking the non-negativity of a polynomial function (Parrilo, 2000). For this reason, we may search for positive semidefinite matrices that are matrix SOS polynomials in Propositions 1–4. To guarantee strict inequalities in the SDP solver, we add $10^{-3}I_{n_z}$ in $P \succeq 0$, $10^{-3}I_{n_x-m}$ in (10), and $10^{-3}I_n$ in (7) and (17). The algorithm is processed in Matlab by use of SOSTOOLS and SDPT3, see (Papachristodoulou et al., 2016). For details on the transformation from SOS to SDP see (Parrilo, 2000).

In the following examples we search for asymptotically stabilizing controllers wrt. two systems using the results of Proposition 1–4. Values presented in this paper have been rounded to three decimals for better visibility.

5.1 Nonlinear Second Order System

We shall test Proposition 1–3 for synthesizing an asymptotically stabilizing constrained controller in the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_1^2 + x_2\\ x_1\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u.$$

First, we pick $z(x) = [x_1, x_1^2 + x_2]^{\top}$, $g_{\perp} = [1,0]$, $\Gamma = I_2$, $\Lambda_1 = [0,1]$ and $\Lambda_2 = [1,0]$. Thus, $\frac{\partial z}{\partial x}$ is unimodular and C1–C2 are satisfied. Then we select $S_h = \mathrm{diag}(9,9)$, $S_1(x) \in \mathbb{R}^{2\times 2}$ with polynomials of degree 2 as elements and test Proposition 1 with Optimization 1 (maximizing $\mathscr{X}_P$), obtaining $\mathscr{X}_P = \{x \in \mathbb{R}^2 \mid \gamma(x) \leq 1\}$, with $\gamma(x) = 9x_1^4 - 0.001x_1^3 + 18x_1^2x_2 + 9.0x_1^2 - 0.001x_1x_2 + 9x_2^2$.

Next, for illustration we select (a minimum bound on P) $x_0^{\top} = [0,2] \in \mathscr{X}_P$ (previously found) and solve (for a new P and F_2) the conditions of Prop. 1 and 3 with Opt. 2 (minimization of S_u) for $S_3(x) \in \mathbb{R}$ a polynomial of degree 6, resulting in $S_u = 100.134$.

Finally, we evaluate Prop. 1 and 3 with Opt. 1 selecting, for instance, $\mathscr{U}_b = \{u \in \mathbb{R} \mid S_u = 11^2 \geq \|u\|^2\}$. The results can be seen in Figure 2, which shows sets $\mathscr{X}_P \subset \mathscr{X}_h$, $\mathscr{X}_P \subset \mathscr{U}_b$, and the phase portrait in x_1–x_2 plane of the closed-loop for 10 extreme initial positions x_0 represented by symbol "$*$". Here all trajectories converge to the origin as expected. In addition, Figure 3 illustrates 5 seconds of respective control actions (calculated with (6)), which are all constrained in $\mathscr{U}_b$. As mentioned in Section 4, we can also use Prop. 2–3. Table 1 shows a comparison between both Propositions for $x_0^{\top} = [0,2] \in \mathscr{X}_P$, $S_u = 11^2$. We conclude that Prop. 2 yields better optimization results.

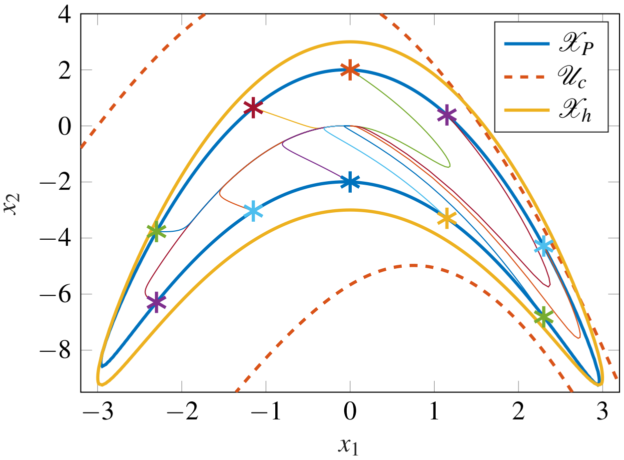

Figura 2. Sets $\mathscr{X}_P$, $\mathscr{X}_h$, $\mathscr{U}_b$, and phase portrait for 10 extreme initial positions.

Tabla 1. Comparison of Prop. 1, 2 in Sec. 5.1.

	Prop. 1, 3	Prop. 2, 3
Opt. 1: det(P)	35.198	41.064
Opt. 2: S_u	100.134	87.345

5.2 Third Order Multiple Input System with AS

Now, we consider a third order system given by

$$\begin{bmatrix}1 & x_1 & 0\\ -x_1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}x_2 - 2x_3 - 2x_1x_2 - 2x_2^2\\ x_1x_2 + u_1\\ x_2x_3 + u_2\end{bmatrix},$$

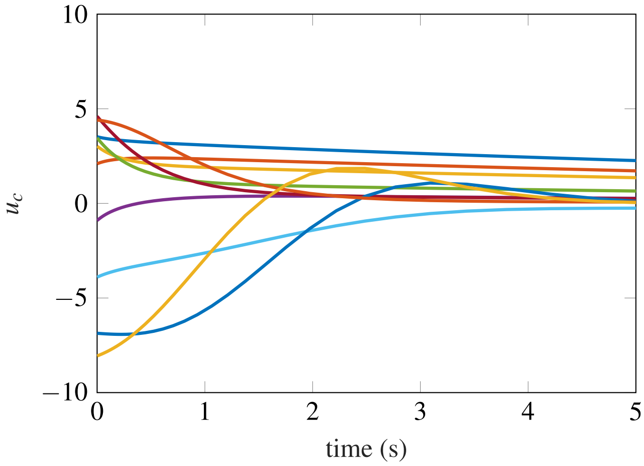

Figura 3. Response of control signal u_b (u_b stays in $\mathscr{U}_b$).

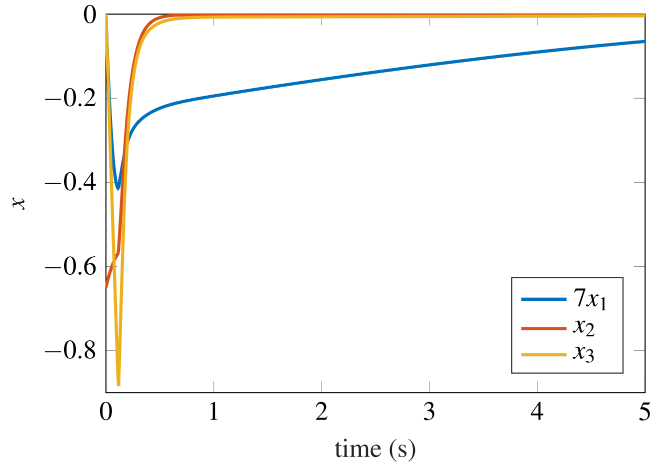

Figura 4. States with $x_0 = [0, -0.65, 0]^\top$ for the third order system in closed loop.

with AS using $u_{\text{sat-n}}$, i.e. $\theta_1 = \theta_2$. In the controller synthesis according to C1–C2, we choose $g_\perp = [1,0,0]$,

$$\Lambda_1^\top = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad \Lambda_2(x) = \begin{bmatrix} x_2 & 0 & 0 \\ 0 & x_3 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & x_1 & 0 \\ -x_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $z(x) = [x_1, x_2, x_2^2 + x_1 x_2 + x_3]^\top$ (unimodular $\frac{\partial z}{\partial x}$). This example is computationally more challenging. Therefore, we select $S_h = \text{diag}(100, 100, 100)$ and use Prop. 2 as fast indicator (that Prop. 1 will work), which is met successfully. Then, we take $S_1 \in \mathbb{R}^{3\times3}$, $F_{20}(x), F_{21}(x) \in \mathbb{R}^{2\times3}$ and $S_3(x) \in \mathbb{R}^{2\times2}$ with polynomials of degree 2 and 4, respectively, and apply Propositions 1, 3, 4 with Opt. 1 for the user defined[7] choice $S_u = \text{diag}(10^2, 8^2)$.

For avoiding excessively large u_x, we constrain each of the constant elements of F_{21} represented by f_{ij} with $|f_{ij}| < 10$. The results are illustrated in Figures 4 and 5. Figure 4 shows the states (x_1 scaled for clarity) in closed-loop under initial condition $[0, -0.65, 0]^\top = x_0 \in \mathscr{X}_P \subset \mathscr{X}_h$. It is clearly seen that all states will converge to the origin. Figure 5 illustrates the first second of $u_{\text{sat-n}}$. Note that u_2 is saturated, obviously, without compromising stability. Furthermore, using Prop. 2 in this system gives worse optimization results, which shows that the selection of the best Proposition (1 or 2) is system dependent.

6. Conclusion

In this paper we provide an algebraic solution for IDA-PBC that is able to resolve the problem of actuator saturation. To this end, we restrict the design to a class of polynomial systems that yield conditions which are solvable with SOS and SDP. The presented algorithm requires the following steps:

S1 Select Λ_1, Λ_2, z, $g_\perp$ and h.

S2 Define S_u and calculate u_b with Propositions 1 (or 2), 3 and Opt. 1 to maximize the volume of $\mathscr{X}_P$. The minimum S_u can also be calculated with Opt. 2.

[7]The minimum S_u can be found similarly as in Example 5.1.

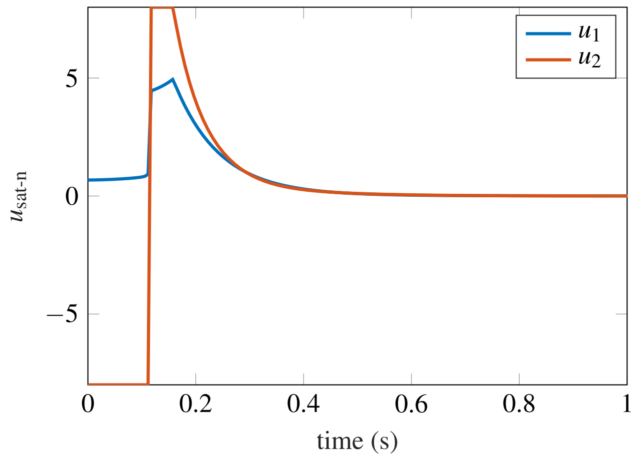

Figura 5. Saturated control action $u_{\text{sat-n}}$.

S3 Compute u_δ with Prop. 4 and P, F_2 found in S2.

S4 Implement the saturation functions $u_{\text{sat-i}}$ or $u_{\text{sat-n}}$.

Additionally, we enjoy features as: no need to solve a PDE, dissipation in design, and one step IDA-PBC. Simulations of two polynomial example systems validate our approach.

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